Principal Stresses as an Eigenvalue Problem

Derivation

Problem Statement

Show that the principal stresses in a two-dimensional stress state can be posed as an eigenvalue problem.

Solution

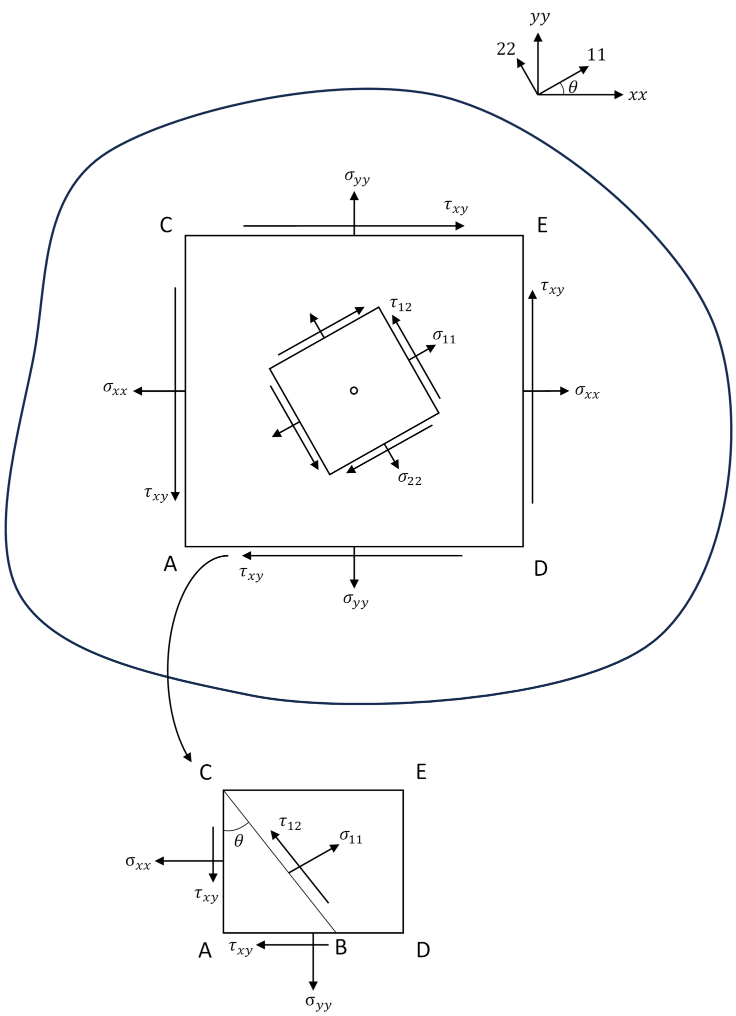

Figure 1: Figure 1: Stresses in x-y and 1-2 coordinate systems

In engineering mechanics, principal stresses are traditionally obtained from stress transformation equations. Linear algebra provides an alternative viewpoint: principal stresses can be interpreted as the eigenvalues of the stress tensor, while the corresponding principal directions are the eigenvectors. The goal of this worked-out problem is to establish this connection starting from basic equilibrium considerations.

Step 1. Stress Transformation and Force Equilibrium

Since principal stresses are associated with specific orientations of a stress element, we first need a relationship between stresses acting in the original x-y coordinate system and stresses acting on a rotated element. These relationships are obtained from force equilibrium.

The transformation of stresses is independent of material properties and depends only on the angle θ between the two co-ordinate systems (Figure 1).

Consider that σ_{xx}, σ_{yy}, and τ_{xy} are the stresses on the rectangular element at a point O in a two-dimensional body (see Figure 1). One now wants to find the stresses σ_{11}, σ_{22}, and τ_{12} on another rectangular element, but at the same point, say O, on the body. To do so, make a cut at an angle θ normal to direction 1. Now the stresses in the 1-2 coordinate system can be related to those in the x-y coordinate system. Summing the forces in direction 1 gives the following

$$\sigma_{11}(\overline{BC}t) - \tau_{xy}(\overline{AB}\ t)\cos\theta - \sigma_{yy}(\overline{AB}\ t)\sin\theta - \tau_{xy}(\overline{AC}\ t)\sin\theta - \sigma_{xx}(\overline{AC}\ t)\cos\theta = 0 \qquad (1)$$

where t is the thickness of the element. Simplifying by dividing by $(\overline{BC}t)$ gives

$$\sigma_{11} = \tau_{xy}\frac{\overline{AB}\ t}{\overline{BC}\ t}\cos\theta + \sigma_{yy}\frac{\overline{AB}\ t}{\overline{BC}\ t}\sin\theta + \tau_{xy}\frac{\overline{AC}\ t}{\overline{BC}\ t}\sin\theta + \sigma_{xx}\frac{\overline{AC}\ t}{\overline{BC}\ t}\cos\theta. \qquad (2)$$

Now,

$$\sin\theta = \frac{\overline{AB}}{\overline{BC}}, \qquad (3)$$

and

$$\cos\theta = \frac{\overline{AC}}{\overline{BC}}, \qquad (4)$$

we get

$$\begin{aligned}\sigma_{11} &= \tau_{xy}\sin\theta\cos\theta + \sigma_{yy}\sin\theta\sin\theta + \tau_{xy}\cos\theta\sin\theta + \sigma_{xx}\cos\theta\cos\theta \\ &= \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta. \qquad (5)\end{aligned}$$

Similarly, by summing the forces in direction 2 gives

$$\tau_{12} = -\sigma_{xx}\sin\theta\cos\theta + \sigma_{yy}\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \qquad (6)$$

By making a cut at an angle, θ, normal to direction 2 as shown in Figure 2 gives

$$\sigma_{22} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta. \qquad (7)$$

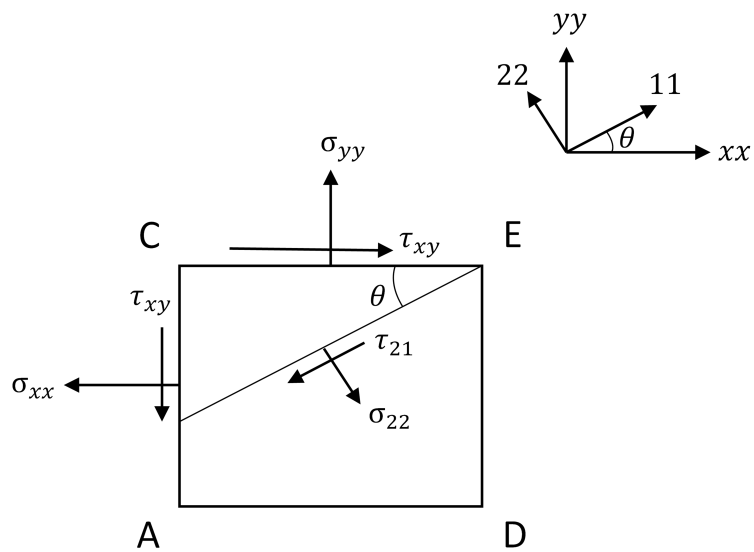

Figure 2: Figure 2: Stresses on another triangular element

Step 2. The Core Condition for Principal Stresses

The principal stresses occur on the element when the shear stresses are zero.

From Equation (5),

$$\sigma_{11} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \qquad (5, repeated)$$

Using trigonometric identities

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad (8)$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad (9)$$

$$2\sin\theta\cos\theta = \sin 2\theta \qquad (10)$$

and substituting Equations (8), (9) and (10) in Equation (5), we get

$$\begin{aligned}\sigma_{11} &= \sigma_{xx}\frac{1+\cos 2\theta}{2} + \sigma_{yy}\frac{1-\cos 2\theta}{2} + \tau_{xy}\sin 2\theta \\ &= \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad (11)\end{aligned}$$

Principal stresses correspond to the extreme values (maximum and minimum) of the normal stress acting on planes passing through the point. Therefore, they can be found by locating stationary points of the normal stress transformation equation. To find the maximum and minimum (the "principal") stresses, we take the derivative of σ_{11} with respect to θ and set it to zero. That is,

$$\frac{d\sigma_{11}}{d\theta} = 0 \qquad (12)$$

Differentiating Equation (11) with respect to θ gives

$$\frac{d\sigma_{11}}{d\theta} = -2\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)\sin 2\theta + 2\tau_{xy}\cos 2\theta \qquad (13)$$

From Equation (6), using prior trigonometric identities given by Equations (8), (9) and 10, we get

$$\begin{aligned}
\tau_{12} &= -\sigma_{xx}\sin\theta\cos\theta + \sigma_{yy}\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \\
&= -\sigma_{xx}\frac{\sin 2\theta}{2} + \sigma_{yy}\frac{\sin 2\theta}{2} + \tau_{xy}\left(\frac{1+\cos 2\theta}{2} - \frac{1-\cos 2\theta}{2}\right) \\
&= -\frac{\sigma_{xx} - \sigma_{yy}}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (14)
\end{aligned}$$

From Equations (13) and (14),

$$\frac{d\sigma_{11}}{d\theta} = 2\tau_{12} \qquad (15)$$

Since

$$\frac{d\sigma_{11}}{d\theta} = 0 \qquad (12, repeated)$$

Then from Equations (12) and (15),

$$\tau_{12} = 0 \qquad (16)$$

Physically, a principal plane is an orientation on which the traction vector acts purely normal to the surface, producing no tangential (shear) component. It has hence been proven that for the derivative of the normal stress to be zero (which determines the maximum or minimum value of normal stresses), the shear stress must be zero. It isn't an assumption; it is a mathematical consequence of seeking the maximum and minimum normal stress.

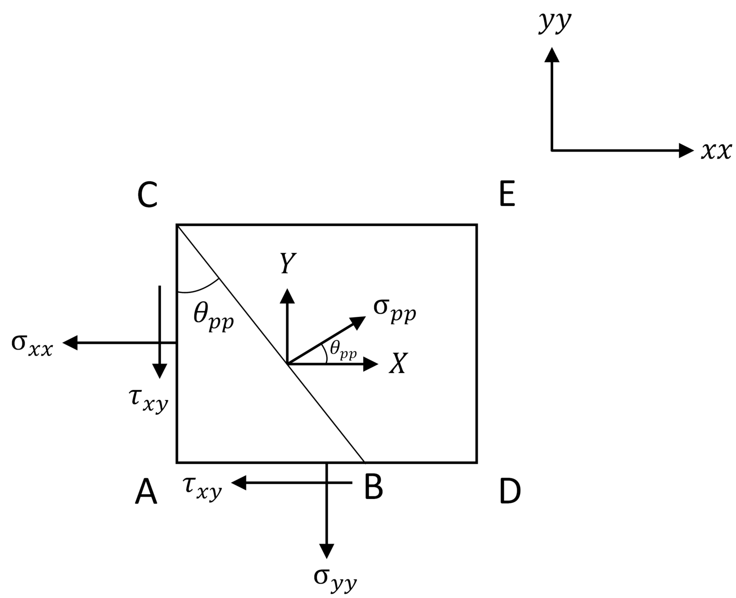

Figure 3: Figure 3: An element with principal stresses

Step 3. Mapping to a Matrix Eigenvalue Problem

Assume that the surface taken at an angle θ_p in the element shown in Figure 3 has no shear stress and only now has the normal stress σ_{pp}. This normal stress σ_{pp} is then one of the principal stresses. The traction associated with the principal stress σ_{pp} can be resolved into components X and Y. Summing the forces in directions x and y, respectively, gives

$$X\left(\overline{BC}\right)t - \sigma_{xx}\left(\overline{AC}\right)t - \tau_{xy}\left(\overline{AB}\right)t = 0 \qquad (17)$$

$$Y\left(\overline{BC}\right)t - \tau_{xy}\left(\overline{AC}\right)t - \sigma_{yy}\left(\overline{AB}\right)t = 0 \qquad (18)$$

Simplifying Equations (17) and (18) by dividing by $(\overline{BC}t)$ gives

$$X = \sigma_{xx}\frac{\overline{AC}}{\overline{BC}} + \tau_{xy}\frac{\overline{AB}}{\overline{BC}} \qquad (19)$$

$$Y = \tau_{xy}\frac{\overline{AC}}{\overline{BC}} + \sigma_{yy}\frac{\overline{AB}}{\overline{BC}} \qquad (20)$$

or

$$X = \sigma_{xx}\cos\theta_p + \tau_{xy}\sin\theta_p \qquad (21)$$

$$Y = \tau_{xy}\cos\theta_p + \sigma_{yy}\sin\theta_p \qquad (22)$$

The stress components X and Y on the BC can also be written in terms of the normal stress σ_{pp}

$$X\left(\overline{BC}\right)t=\sigma_{pp}\left(\overline{BC}\right)t\cos\theta_p \tag{23}$$

$$Y\left(\overline{BC}\right)t=\sigma_{pp}\left(\overline{BC}\right)t\sin\theta_p \tag{24}$$

Simplifying by dividing by ($\overline{BC}t$) gives

$$X=\sigma_{pp}\cos\theta_p \tag{25}$$

$$Y=\sigma_{pp}\sin\theta_p \tag{26}$$

Hence, from Equations (21), (22), (25) and (26)

$$\sigma_{pp}\cos\theta_p=\sigma_{xx}\cos\theta_p+\tau_{xy}\sin\theta_p \tag{27}$$

$$\sigma_{pp}\sin\theta_p=\tau_{xy}\cos\theta_p+\sigma_{yy}\sin\theta_p \tag{28}$$

Taking all variables to the left hand side in Equations (27) and (28) gives

$$(\sigma_{xx}-\sigma_{pp})\cos\theta_p+\tau_{xy}\sin\theta_p=0 \tag{29}$$

$$\tau_{xy}\cos\theta_p+(\sigma_{yy}-\sigma_{pp})\sin\theta_p=0 \tag{30}$$

In terms of matrices, Equations (29) and (30) can be rewritten as

$$\begin{bmatrix}\sigma_{xx}-\sigma_{pp} & \tau_{xy}\\ \tau_{xy} & \sigma_{yy}-\sigma_{pp}\end{bmatrix}\begin{bmatrix}\cos\theta_p\\ \sin\theta_p\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \tag{31}$$

$$\left(\begin{bmatrix}\sigma_{xx} & \tau_{xy}\\ \tau_{xy} & \sigma_{yy}\end{bmatrix}-\begin{bmatrix}0 & -\sigma_{pp}\\ -\sigma_{pp} & 0\end{bmatrix}\right)\begin{bmatrix}\cos\theta_p\\ \sin\theta_p\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \tag{32}$$

$$\left(\begin{bmatrix}\sigma_{xx} & \tau_{xy}\\ \tau_{xy} & \sigma_{yy}\end{bmatrix}-\sigma_{pp}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}\cos\theta_p\\ \sin\theta_p\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \tag{33}$$

Equation (33) can be written as

$$(A-\sigma_{pp}I)\,n=0 \tag{34}$$

where

$$[A]=\begin{bmatrix}\sigma_{xx} & \tau_{xy}\\ \tau_{xy} & \sigma_{yy}\end{bmatrix} \tag{35}$$

is the stress tensor matrix,

$$I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \tag{36}$$

is the identity matrix,

$$n = \begin{bmatrix} \cos\theta_p \\ \sin\theta_p \end{bmatrix} \qquad (37)$$

is the directional unit normal vector. The vector n is a unit normal vector defining the orientation of the plane on which the principal stress acts. Thus, the eigenvector does not represent a displacement or force; it represents a geometric direction.

Equation (34) can also be written as

$$An = \sigma_{pp} n \qquad (38)$$

Equation (38) has a simple physical interpretation. The stress tensor maps a unit normal vector n into the traction vector acting on the plane whose normal is n. For most planes, the traction vector has both normal and shear components and is therefore not parallel to n. On a principal plane, however, the traction vector is aligned with the normal vector. The stress tensor simply scales the vector n by the factor σ_{pp}. This is precisely the definition of an eigenvalue-eigenvector pair. Equation (38) is an eigenvalue problem, where σ_{pp} is the eigenvalue. Finding the principal stresses amounts to finding the eigenvalues of the matrix $\begin{bmatrix} A \end{bmatrix}$, and finding the principal directions amounts to calculating its eigenvectors.

Since the stress tensor is symmetric, its eigenvalues are real, and its eigenvectors are orthogonal.

Step 4. Finding the Principal Stresses and the Direction

The only way the set of equations given by Equation (34) can have a non-trivial solution for n is if

$$\det\left(A - \sigma_{pp} I\right) = 0 \qquad (39)$$

Why does n need to be non-trivial? Because the n vector cannot be trivial, as $\cos\theta_p$ and $\sin\theta_p$ cannot both be zero because $\cos^2\theta_p + \sin^2\theta_p = 1$. The eigenvectors corresponding to the two eigenvalues would then give the direction of the principal stresses.

Conclusion

The principal stress problem is fundamentally an eigenvalue problem. The stress tensor maps the normal vector of a plane into the traction vector acting on that plane. Principal planes are those special orientations for which the traction vector remains parallel to the plane normal. The corresponding scaling factors are the principal stresses (eigenvalues), and the associated directions are the principal directions (eigenvectors).